

On the low energy limit of one loop photon-graviton amplitudes

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Abstract: We present first results of a systematic study of the structure of the low energy limit of the one-loop photon-graviton amplitudes induced by massive scalars and spinors. Our main objective is the search of KLT-type relations where effectively two photons merge into a graviton. We find such a relation at the graviton-photon-photon level. We also derive the diffeomorphism Ward identity for the 1PI one graviton - N photon amplitudes.

1 Introduction

In recent years, much effort has been devoted to the study of the structure of graviton amplitudes. This was largely due to developments in string theory, which led to the prediction that such amplitudes should be much more closely related to gauge theory amplitudes than one would suspect by comparing the Lagrangians or Feynman rules of gravitational and gauge theories. Specifically, the Kawai-Lewellen-Tye (KLT) relations in string theory imply that graviton amplitudes should be “squares” of gauge theory amplitudes [1, 2, 3, 4, 5]. String theory was also instrumental in providing guiding principles to develop new powerful techniques for the computation of graviton amplitudes [6, 7, 8, 9]. Additional motivation comes from the possible finiteness of $N = 8$ supergravity (see [10, 11] and refs. therein).

This work was largely confined to the case of massless on-shell amplitudes, for which particularly efficient computation methods are available. Relatively little seems to have been done on amplitudes involving the interaction of gravitons with massive matter. At the tree level, there are some classical results on amplitudes involving gravitons [12, 13]. More recently, the tree-level Compton-type amplitudes involving gravitons and spin zero, half and one particles were computed [14] to verify another remarkable factorization property [15] of the graviton-graviton scattering amplitudes in terms of the photonic Compton amplitudes. At the level of one (matter) loop, to the best of our knowledge so far the only mixed graviton - gauge boson amplitudes that have been computed are the graviton-photon-photon vertex [16, 17, 18, 19, 20], its nonabelian generalization [21], and the related amplitude for photon-graviton conversion in an external field [22, 23, 24, 25, 26].

We believe that new insight into the structural relations between photon and graviton amplitudes might be obtained by studying the N graviton amplitudes involving a massive loop, and more generally the mixed one-loop graviton-photon amplitudes. Generally, massive one-loop N - point amplitudes are significantly more difficult to compute than massless ones; on the other hand, their large mass limit, which is also the low energy limit, is accessible through the effective action. For the prototypical case, the QED N - photon amplitude, the information on this low energy limit is contained in the Euler-Heisenberg Lagrangian (“EHL”) [27]. We recall the standard proper time representation of this effective Lagrangian:

$$\mathcal{L}_{\text{spin}}^{(EH)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[\frac{(eaT)(ebT)}{\tanh(eaT) \tan(ebT)} - \frac{e^2}{3} (a^2 - b^2) T^2 - 1 \right]. \quad (1.1)$$

Here T is the proper-time of the loop fermion, m its mass, and a, b are the two Maxwell field invariants, related to \mathbf{E}, \mathbf{B} by $a^2 - b^2 = B^2 - E^2$, $ab = \mathbf{E} \cdot \mathbf{B}$. The subtraction terms implement the renormalization of charge and vacuum energy. The analogous representation for scalar QED, due to Weisskopf [28], is

$$\mathcal{L}_{\text{scal}}^{(EH)}(F) = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[\frac{(eaT)(ebT)}{\sinh(eaT) \sin(ebT)} + \frac{1}{6} (a^2 - b^2) T^2 - 1 \right] \quad (1.2)$$

After expanding the EHL in powers of the field invariants, it is straightforward to obtain the low energy (= large mass) limit of the N photon amplitudes from the terms in this expansion involving N powers of the field. The result of this procedure can be expressed quite concisely [29]:

$$\begin{aligned} A_{\text{spin}}[\varepsilon_1^+; \dots; \varepsilon_K^+; \varepsilon_{K+1}^-; \dots; \varepsilon_N^-] &= -\frac{m^4}{8\pi^2} \left(\frac{2ie}{m^2} \right)^N (N-3)! \\ &\quad \times \sum_{k=0}^K \sum_{l=0}^{N-K} (-1)^{N-K-l} \frac{\mathcal{B}_{k+l} \mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!} \chi_K^+ \chi_{N-K}^-, \\ A_{\text{scal}}[\varepsilon_1^+; \dots; \varepsilon_K^+; \varepsilon_{K+1}^-; \dots; \varepsilon_N^-] &= \frac{m^4}{16\pi^2} \left(\frac{2ie}{m^2} \right)^N (N-3)! \\ &\quad \times \sum_{k=0}^K \sum_{l=0}^{N-K} (-1)^{N-K-l} \frac{(1-2^{1-k-l})(1-2^{1-N+k+l}) \mathcal{B}_{k+l} \mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!} \chi_K^+ \chi_{N-K}^-. \end{aligned} \quad (1.3)$$

Here the superscripts \pm refer to circular polarizations, and the \mathcal{B}_k are Bernoulli numbers. The invariants χ_K^\pm are written, in spinor helicity notation (our spinor helicity conventions follow [30]),

$$\begin{aligned}
\chi_K^+ &= \frac{(\frac{K}{2})!}{2^{\frac{K}{2}}} \left\{ [12]^2 [34]^2 \cdots [(K-1)K]^2 + \text{all permutations} \right\}, \\
\chi_K^- &= \frac{(\frac{K}{2})!}{2^{\frac{K}{2}}} \left\{ \langle 12 \rangle^2 \langle 34 \rangle^2 \cdots \langle (K-1)K \rangle^2 + \text{all permutations} \right\}.
\end{aligned}
\tag{1.4}$$

For the case of the “maximally helicity-violating” (MHV) amplitudes, which have “all +” or “all -” helicities, eqs. (1.3) simplify (using Bernoulli number identities) to

$$A_{\text{scal}}[k_1, \varepsilon_1^\pm; \dots; k_N, \varepsilon_N^\pm] = -\frac{(2e)^N}{(4\pi)^2 m^{2N-4}} \frac{\mathcal{B}_N}{N(N-2)} \chi_N^\pm \tag{1.5}$$

$$A_{\text{spin}}[k_1, \varepsilon_1^\pm; \dots; k_N, \varepsilon_N^\pm] = -2A_{\text{scal}}[k_1, \varepsilon_1^\pm; \dots; k_N, \varepsilon_N^\pm] \tag{1.6}$$

This relation (1.6) is actually true also away from the low-energy limit, and can be explained by the fact that the MHV amplitudes correspond to a self-dual background, in which the Dirac operator has a quantum-mechanical supersymmetry [31]. For this MHV case eqs. (1.3) have also been generalized to the two-loop level [32].

One of the long-term goals of the present line of work is to obtain a generalization of (1.3) to the case of the mixed N - photon / M - graviton amplitudes. As a first step, in [33] the EHL (1.1) and its scalar analogue were generalized to the effective actions corresponding to the low energy one graviton - N photon amplitudes. These are the effective actions in Einstein-Maxwell theory with a scalar or spinor loop, computed to all orders in the electromagnetic field strength, to leading order in the curvature, and also including terms where the curvature tensor gets replaced by two covariant derivatives. They were obtained in [33] in terms of two-parameter integrals (a similar result was found in [34]). Although expanding them out in powers of the field invariants is straightforward in principle, in practice (contrary to the QED case) it is a very laborious procedure. In [33] this was done only at the F^2 level, as a check of consistency with previous results in the literature. In particular, the F^2 part for the spinor loop was shown to coincide, up to total derivative terms, with the effective Lagrangian obtained in the seminal work of Drummond and Hathrell [16],

$$\mathcal{L}_{\text{spin}}^{(DH)} = \frac{e^2}{180(4\pi)^2 m^2} \left(5R F_{\mu\nu}^2 - 26R_{\mu\nu} F^{\mu\alpha} F^\nu{}_\alpha + 2R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2 \right) \quad (1.7)$$

(see [33] for our gravity conventions). The corresponding form of the effective Lagrangian for the scalar loop (using the same operator basis) is [33]

$$\mathcal{L}_{\text{scal}}^{(DH)} = \frac{e^2}{180(4\pi)^2 m^2} \left[15\left(\xi - \frac{1}{6}\right) R F_{\mu\nu}^2 - 2R_{\mu\nu} F^{\mu\alpha} F^\nu{}_\alpha - R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} + 3(\nabla^\alpha F_{\alpha\mu})^2 \right]. \quad (1.8)$$

(the parameter ξ refers to a non-minimal coupling of the scalar). In [35] two of the present authors presented the next order in the expansion of the effective Lagrangians obtained in [33] in powers of the field strength, i.e. the terms of order RF^4 (there are no order RF^3 terms for parity reasons).

The purpose of the present letter is twofold. First, we will use the previously obtained effective actions at the RF^2 level to compute the low energy limits of the one graviton - two photon amplitudes with a scalar and spinor loop, and show that they relate to the four photon amplitudes in a KLT - like way. Second, as a preparation for the study of the higher-point cases we will derive the Ward identities for the one graviton - N photon 1PI amplitudes in general.

2 Ward identities for the 1PI one graviton – N photon amplitudes

We derive the relevant Ward identities, generalizing the discussion in [24]. There are two types of Ward identities, those derived from gauge invariance and those that follow from general coordinate invariance.

Gauge transformations are defined by

$$\delta_G A_\mu = \partial_\mu \lambda, \quad \delta_G g_{\mu\nu} = 0 \quad (2.1)$$

with an arbitrary local parameter λ . Then gauge invariance of the effective action

$$\delta_G \Gamma[g, A] = 0 \quad (2.2)$$

implies that

$$\nabla_\mu \left(\frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A_\mu} \right) = 0 . \quad (2.3)$$

Similarly, infinitesimal reparametrizations are given by

$$\delta_R A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu , \quad \delta_R g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (2.4)$$

with arbitrary local parameters ξ^μ . The invariance of the effective action

$$\delta_R \Gamma[g, A] = 0 \quad (2.5)$$

now implies

$$\nabla_\mu \left(\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A_\mu} A^\nu \right) - \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A_\mu} \nabla^\nu A_\mu = 0 . \quad (2.6)$$

The Ward identities thus obtained can be combined and written more conveniently using standard tensor calculus as follows

$$\partial_\mu \frac{\delta \Gamma}{\delta A_\mu} = 0 , \quad (2.7)$$

$$2\partial_\mu \frac{\delta \Gamma}{\delta g_{\mu\nu}} + \frac{\delta \Gamma}{\delta A_\mu} \partial_\mu A^\nu + \Gamma_{\mu\lambda}^\nu \left(2 \frac{\delta \Gamma}{\delta g_{\mu\lambda}} + \frac{\delta \Gamma}{\delta A_\mu} A^\lambda \right) - \frac{\delta \Gamma}{\delta A_\mu} \nabla^\nu A_\mu = 0 . \quad (2.8)$$

We remark that, alternatively, the Ward identities from gauge invariance can be used to simplify the Ward identities from reparametrizations. In fact, an infinitesimal reparametrization can be written as

$$\delta_R A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu = \partial_\mu (\xi^\nu A_\nu) + \xi^\nu F_{\nu\mu} \quad (2.9)$$

$$\delta_R g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (2.10)$$

where the first term in the last rule for A_μ can be interpreted as a gauge transformation. The invariance of the effective action $\delta_R \Gamma[g, A] = 0$ now implies (making use of $\delta_G \Gamma[g, A] = 0$ as well)

$$\nabla_\mu \left(\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}} \right) + \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A_\mu} F_\mu{}^\nu = 0 \quad (2.11)$$

i.e.

$$2\partial_\mu \frac{\delta\Gamma}{\delta g_{\mu\nu}} + 2\Gamma^\nu_{\mu\lambda} \frac{\delta\Gamma}{\delta g_{\mu\lambda}} + \frac{\delta\Gamma}{\delta A_\mu} F_\mu{}^\nu = 0 \quad (2.12)$$

which of course is equivalent to (2.8).

Now we consider the special case of the correlation function of one graviton and N photons in flat space.

$$\Gamma_{(x_0, x_1, \dots, x_N)}^{\mu\nu, \alpha_1 \dots \alpha_N} \equiv \frac{\delta^{N+1}\Gamma}{\delta g_{\mu\nu}(x_0) \delta A_{\alpha_1}(x_1) \dots \delta A_{\alpha_N}(x_N)} \Big|_{g_{\mu\nu}=\eta_{\mu\nu}, A_{\alpha_1}=A_{\alpha_2}=\dots=A_{\alpha_N}=0} \quad (2.13)$$

Taking appropriate functional derivatives on the general Ward identities (2.7), (2.12) we obtain the gauge Ward identities

$$\partial_{\alpha_i}^{(x_i)} \Gamma_{(x_0, x_1, \dots, x_N)}^{\mu\nu, \alpha_1 \dots \alpha_N} = 0, \quad i = 1, \dots, N, \quad (2.14)$$

and the gravitational Ward identities

$$\begin{aligned} & \sum_{i=1}^N \frac{\delta^N \Gamma}{\delta A_\mu(x_0) \delta A_{\alpha_1}(x_1) \dots \widehat{\delta A_{\alpha_i}(x_i)} \dots \delta A_{\alpha_N}(x_N)} \Big| (\delta_\mu^{\alpha_i} \partial_{(x_i)}^\nu - \eta^{\alpha_i \nu} \partial_\mu^{(x_i)}) \delta^D(x_0 - x_i) \\ & + 2\partial_\mu^{(x_0)} \Gamma_{(x_0, x_1, \dots, x_N)}^{\mu\nu, \alpha_1 \dots \alpha_N} = 0, \end{aligned} \quad (2.15)$$

where the “hat” means omission.

Fourier transforming the identities (2.14), (2.15) to momentum space

$$\int dx_0 dx_1 \dots dx_N e^{ik_0 x_0 + \dots + ik_N x_N} \Gamma_{(x_0, \dots, x_N)} = (2\pi)^D \delta(k_0 + \dots + k_N) \Gamma[k_0, \dots, k_N] \quad (2.16)$$

they turn into

$$k_{i\alpha_i} \Gamma^{\mu\nu, \alpha_1 \dots \alpha_N}[k_0, \dots, k_N] = 0, \quad i = 1, \dots, N, \quad (2.17)$$

$$\begin{aligned} & 2k_{0\mu} \Gamma^{\mu\nu, \alpha_1 \dots \alpha_N}[k_0, \dots, k_N] + \sum_{i=1}^N \Gamma^{\mu\alpha_1 \dots \widehat{\alpha_i} \dots \alpha_N}[k_0 + k_i, k_1, \dots, \widehat{k_i}, \dots, k_N] (\delta_\mu^{\alpha_i} k_i^\nu - \eta^{\alpha_i \nu} k_{i\mu}) \\ & = 0. \end{aligned} \quad (2.18)$$

Thus the gauge Ward identity is transversal as in QED, while the gravitational Ward identity relates the one graviton – N photon amplitude to the pure N photon amplitudes.

3 The graviton-photon-photon amplitude

We proceed to the study of the on-shell graviton-photon-photon amplitude induced by a scalar or spinor loop (see figure 1).

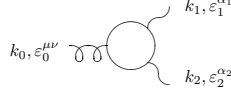


Figure 1: Graviton – photon – photon diagram

First, let us remark that this amplitude does not exist at tree level for the fully on-shell case. The covariantized Maxwell term in the action of Einstein-Maxwell theory

$$S[g, A] = \int d^D x \sqrt{g} \left(\frac{1}{\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad (3.1)$$

contains a graviton-photon-photon vertex, and this vertex with one photon leg off-shell is responsible for the well-known process of photon-graviton conversion in an electromagnetic field [22, 23, 24, 25, 26]. However, it vanishes with all legs on-shell (let us also remark that, fully off-shell, this vertex provides already an example for the non-trivialness of the gravitational Ward identity (2.18)).

The low-energy limit of the one-loop amplitudes is readily obtained from the Drummond-Hathrell form of the effective Lagrangians, (1.7) resp. (1.8). In the on-shell case, the R , $R_{\mu\nu}$ and $(\nabla^\alpha F_{\alpha\mu})^2$ terms all vanish. The remaining term can, after the usual procedure of taking the Fourier transform and then truncating to lowest order in momenta [24], be written as

$$\begin{aligned} R_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu} &= \varepsilon_{0\mu\nu} \Gamma^{\mu\nu;\alpha\beta} \varepsilon_{1\alpha} \varepsilon_{2\beta} \\ \Gamma^{\mu\nu;\alpha\beta} &= 4 \left[-k_1^{(\mu} k_2^{\nu)} k_0^\alpha k_0^\beta + \eta^{\alpha(\mu} k_2^{\nu)} k_0^\beta k_0 \cdot k_1 \right. \\ &\quad \left. + \eta^{\beta(\mu} k_1^{\nu)} k_0^\alpha k_0 \cdot k_2 - \eta^{\alpha(\mu} \eta^{\nu)\beta} k_0 \cdot k_1 k_0 \cdot k_2 \right] \end{aligned} \quad (3.2)$$

At the $N = 2$ level the purely photonic terms in the gravitational Ward identity (2.18) vanish on-shell. Thus this identity holds in its usual form $k_{0\mu} \Gamma^{\mu\nu;\alpha\beta} = 0$, as can be checked with (3.2).

Proceeding to the helicity decomposition of the amplitude, using a factorized graviton polarization tensor as usual,

$$\begin{aligned}\varepsilon_{0\mu\nu}^{++}(k_0) &= \varepsilon_{0\mu}^+(k_0)\varepsilon_{0\nu}^+(k_0), \\ \varepsilon_{0\mu\nu}^{--}(k_0) &= \varepsilon_{0\mu}^-(k_0)\varepsilon_{0\nu}^-(k_0),\end{aligned}\tag{3.3}$$

together with a judicious choice of the reference momenta one can easily show that, of the six components of this amplitude, only the MHV ones are nonvanishing ¹ :

$$\begin{aligned}\varepsilon_{0\mu\nu}^{++}\Gamma^{\mu\nu;\alpha\beta}\varepsilon_{1\alpha}^+\varepsilon_{2\beta}^+ &= -[01]^2[02]^2 \\ \varepsilon_{0\mu\nu}^{--}\Gamma^{\mu\nu;\alpha\beta}\varepsilon_{1\alpha}^-\varepsilon_{2\beta}^- &= -\langle 01\rangle^2\langle 02\rangle^2\end{aligned}\tag{3.4}$$

Including the prefactors in (1.7),(1.8), and restoring the coupling constants, we obtain the final result,

$$\begin{aligned}A_{\text{spin}}^{(++;++)} &= -\frac{\kappa e^2}{90(4\pi)^2 m^2} [01]^2 [02]^2 \\ A_{\text{spin}}^{(--;--)} &= -\frac{\kappa e^2}{90(4\pi)^2 m^2} \langle 01\rangle^2 \langle 02\rangle^2\end{aligned}\tag{3.5}$$

Here the first upper index pair refers to the graviton polarization, and κ is the gravitational coupling constant. Moreover, those components fulfill the MHV relation (1.6),

$$\begin{aligned}A_{\text{spin}}^{(++;++)} &= (-2) A_{\text{scal}}^{(++;++)} \\ A_{\text{spin}}^{(--;--)} &= (-2) A_{\text{scal}}^{(--;--)}\end{aligned}\tag{3.6}$$

¹It should be mentioned that even for these components the right hand sides will vanish after taking into account that for a massless three-point amplitude energy-momentum conservation forces collinearity of the three momenta. However, this is a low-point kinematic accident and not relevant for our structural investigation.

Also, these graviton-photon-photon amplitudes relate to the (low energy) four photon amplitudes in the following way: From (1.3), (1.4) the only non-vanishing components of those are

$$\begin{aligned}
A^{++++}[k_1, k_2, k_3, k_4] &\sim [12]^2[34]^2 + [13]^2[24]^2 + [14]^2[23]^2 \\
A^{++--}[k_1, k_2, k_3, k_4] &\sim [12]^2\langle 34 \rangle^2 \\
A^{----}[k_1, k_2, k_3, k_4] &\sim \langle 12 \rangle^2\langle 34 \rangle^2 + \langle 13 \rangle^2\langle 24 \rangle^2 + \langle 14 \rangle^2\langle 23 \rangle^2
\end{aligned} \tag{3.7}$$

Replacing $k_1 \rightarrow k_0, k_2 \rightarrow k_0$ in the 4 photon amplitudes, the middle one of these three components becomes zero, and the remaining ones become proportional to the corresponding components of (3.5):

$$\begin{aligned}
A^{++++}[k_0, k_0, k_3, k_4] &\sim 2[03]^2[04]^2 \sim A^{++;++}[k_0, k_3, k_4] \\
A^{----}[k_0, k_0, k_3, k_4] &\sim 2\langle 03 \rangle^2\langle 04 \rangle^2 \sim A^{--;--}[k_0, k_3, k_4]
\end{aligned} \tag{3.8}$$

Thus in all cases one finds the same proportionality, namely

$$A_{\text{scal,spin}}^{\pm\pm;\pm\pm}[k_0, k_1, k_2] = \frac{1}{12} \frac{\kappa}{e^2} m^2 A_{\text{scal,spin}}^{\pm\pm\pm\pm}[k_0, k_0, k_1, k_2] \tag{3.9}$$

Effectively two photons have merged to form a graviton, clearly a result in the spirit of the KLT relations.

4 Conclusions

We have shown here that, in the low energy limit, the one-loop graviton-photon-photon amplitudes in Einstein-Maxwell theory coupled to scalars or spinors relate to a coincidence limit of the QED four photon amplitudes. This is due to a KLT like factorization of the graviton into two photons, and raises the possibility that the low energy M graviton – N photon amplitudes may be derivable from the $N + 2M$ photon amplitudes. However, the three-point amplitude is rather special in this context due to the absence of one-particle reducible contributions. The inhomogeneity of the gravitational Ward identity (2.18) leads one to expect that, starting from the one graviton – four photon level, the 1PI one graviton - N photon amplitudes will not be transversal in the graviton indices, so that a relation with the purely

photonic amplitudes can exist only for the full amplitudes. The calculation of the one graviton – four photon amplitude is in progress.

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